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LETTER TO THE EDITOR

Consistent multiparameter quantisation of GL(n)

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Abstract. We describe a manifold of quantum group structures on the vector space of the universal enveloping algebra of $\mathbf{gl}(n)$ and on its dual, the space of polynomials in n^2 variables. The dimension of the manifold is $(n^2-n+2)/2$.

Manin [1, 2] has investigated a family of quantum groups, deformations of the algebra of polynomial functions on GL(n), depending on $N=n(n-1)/2$ parameters. He found, however, that in this family only the usual one-parameter deformations were consistent with functional independence of the generators in the sense of the Poincaré-Birkhoff-Witt theorem, so that the remaining structures were defined on a smaller space than that of the algebra of functions. This letter is devoted to the construction of an $(N+1)$-parameter family of quantum deformations of GL(n) in which the consistency condition is satisfied for all values of the parameters. The algebra is presented both as an algebra generated by n^2 independent non-commuting matrix elements, with matrix comultiplication, and in the dual form as a deformation G of the universal enveloping algebra of $\mathbf{gl}(n)$. We start with the latter form; later we construct the algebra G^* of non-commuting matrix elements by considering the fundamental representation of G. Finally we consider $n=2$ and some other special cases, and comment on the relation of these quantum groups to the Yang-Baxter equation.

Let A be the polynomial algebra generated by $x_1, \ldots, x_n$ with relations

$$x_j x_i = q_{ij} x_i x_j \tag{1}$$

where q_{ij} are c-numbers satisfying $q_{ii}=1$ and $q_{ji}=q_{ij}^{-1}$. It will be convenient to define

$$p_{ij} = \begin{cases} q_{ij} & \text{if } i<j \\ 1 & \text{if } i \geqslant j \end{cases}$$

and to write (1) as

$$p_{ji} x_j x_i = p_{ij} x_i x_j \tag{2}$$

(no summation convention; throughout this letter all summations will be explicitly marked).

The algebra A is spanned by the monomials

$$\boldsymbol{x}^{\boldsymbol{r}} = x_n^{r_n} \ldots x_1^{r_1}$$

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with $\boldsymbol{r}=(r_1,\ldots,r_n)\in\mathbb{N}^n$. We define operators $D_i: A\to A$ and $X_{ij}: A\to A$ (for $i\neq j$) by

$$D_i(\boldsymbol{x}^{\boldsymbol{r}})=r_i\boldsymbol{x}^{\boldsymbol{r}} \tag{3}$$

$$X_{ij}(\boldsymbol{x}^{\boldsymbol{r}})=[r_i]_u\boldsymbol{x}^{\boldsymbol{r}-\boldsymbol{e}_i+\boldsymbol{e}_j} \tag{4}$$

where $\boldsymbol{e}_i$ is the elementary vector whose jth component is δ_{ij}, and

$$[x]_u=\frac{u^x-u^{-x}}{u-u^{-1}} \tag{5}$$

u being a further independent parameter. These operators satisfy

$$D_i(\boldsymbol{x}^{\boldsymbol{r}}\boldsymbol{x}^{\boldsymbol{s}})=(D_i\boldsymbol{x}^{\boldsymbol{r}})\boldsymbol{x}^{\boldsymbol{s}}+\boldsymbol{x}^{\boldsymbol{r}}(D_i\boldsymbol{x}^{\boldsymbol{s}}) \tag{6}$$

and

$$X_{ij}(\boldsymbol{x}^{\boldsymbol{r}}\boldsymbol{x}^{\boldsymbol{s}})=(X_{ij}\boldsymbol{x}^{\boldsymbol{r}})\frac{u^{s_i}\boldsymbol{a}_j^{\boldsymbol{s}}}{\boldsymbol{a}_i^{\boldsymbol{s}}}\boldsymbol{x}^{\boldsymbol{s}}+\frac{u^{-r_i}\boldsymbol{b}_j^{\boldsymbol{r}}}{\boldsymbol{b}_i^{\boldsymbol{r}}}\boldsymbol{x}^{\boldsymbol{r}}(X_{ij}\boldsymbol{x}^{\boldsymbol{s}}) \tag{7}$$

where

$$\boldsymbol{a}_i^{\boldsymbol{s}}=p_{i1}^{s_1}\ldots p_{in}^{s_n}\qquad\qquad \boldsymbol{b}_i^{\boldsymbol{r}}=p_{1i}^{r_1}\ldots p_{ni}^{r_n}. \tag{8}$$

Thus D_i and X_{ij} are generalised (twisted) derivations of A with the coproducts

$$\Delta(D_i)=D_i\otimes 1+1\otimes D_i \tag{9}$$

and

$$\Delta(X_{ij})=X_{ij}\otimes u^{D_i}A_i^{-1}A_j+u^{-D_i}B_i^{-1}B_j\otimes X_{ij} \tag{10}$$

where A_i and B_i are the operators (functions of $D_1,\ldots,D_n$) whose eigenvalues are given by (8).

Equation (7) remains true if u is replaced by u^{-1}; thus we could also choose the coproduct obtained from (10) by changing u to u^{-1}.

Let

$$E_i=X_{i,i+1}\qquad\qquad F_i=X_{i+1,i}.$$

Then D_i, E_i and F_i satisfy

$$\begin{aligned}&[D_i,D_j]=0\\&[D_i,E_j]=(-\delta_{ij}+\delta_{i,j+1})E_j\\&[D_i,F_j]=(\delta_{ij}-\delta_{i,j+1})F_j\\&[E_i,F_j]=[D_{i+1}-D_i]_u\delta_{ij}\end{aligned} \tag{11}$$

$$\begin{aligned}&E_{i\pm1}E_i^2+E_i^2E_{i\pm1}=(u+u^{-1})E_iE_{i\pm1}E_i\\&F_{i\pm1}F_i^2+F_i^2F_{i\pm1}=(u+u^{-1})F_iF_{i\pm1}F_i\end{aligned} \tag{12}$$

and

$$[E_i,E_j]=[F_i,F_j]=0\qquad\text{if}\qquad |i-j|\geqslant 2. \tag{13}$$

In view of (9) and (10) and the remark following them, we choose coproducts

$$\begin{aligned}&\Delta(D_i)=D_i\otimes 1+1\otimes D_i\\&\Delta(E_i)=E_i\otimes u^{D_i}R_i+u^{-D_i}C_i\otimes E_i\\&\Delta(F_i)=F_i\otimes u^{-D_{i+1}}R_i^{-1}+u^{D_{i+1}}C_i^{-1}\otimes F_i\end{aligned} \tag{14}$$

where

$$R_i = A_i^{-1} A_{i+1} = \prod_{k=1}^{n} \left[\frac{p_{i+1,k}}{p_{ik}} \right]^{D_k}$$

$$C_i = B_i^{-1} B_{i+1} = \prod_{k=1}^{n} \left[\frac{p_{k,i+1}}{p_{ki}} \right]^{D_k}.$$

The commutators (11) give rise to the following relations between D_i, E_i, F_i, R_i and C_i:

$$\begin{aligned} E_i f(D_i) &= f(D_i + 1) E_i \qquad & E_i f(D_{i+1}) &= f(D_{i+1} - 1) E_i \\ F_i f(D_i) &= f(D_i - 1) F_i \qquad & F_i f(D_{i+1}) &= f(D_{i+1} + 1) F_i \end{aligned} \tag{15}$$

for any function f; and

$$\begin{aligned} C_j E_i &= s_{ij} E_i C_j \qquad & C_j F_i &= s_{ij}^{-1} F_i C_j \\ R_j E_i &= s_{ji} E_i R_j \qquad & R_j F_i &= s_{ji}^{-1} F_i R_j \end{aligned} \tag{16}$$

where

$$s_{ij} = \frac{p_{ij} p_{i+1,j+1}}{p_{i,j+1} p_{i+1,j}}.$$

Using these, it is straightforward to verify that the algebra G generated by D_i, E_i and F_i, with relations (11)-(13) and comultiplication (14), is a bialgebra [2-4], i.e. that the coproducts (14) are compatible with the relations (11)-(13). It becomes a Hopf algebra when furnished with the usual co-unit and the antipode

$$\begin{aligned} S(D_i) &= -D_i \\ S(E_i) &= -u^{-1} R_i^{-1} E_i C_i^{-1} \\ S(F_i) &= -u R_i F_i C_i. \end{aligned} \tag{17}$$

This construction could also be presented in a harmonic oscillator formalism, the monomials $x^r = x_n^{r_n} \ldots x_1^{r_1}$ being replaced by states $|r_n \ldots r_1\rangle$ of n u-deformed oscillators [5-7] with the ith oscillator in its r_ith excited state. The coordinates x_i then correspond to raising operators for the oscillators. If we now make the usual identification of the states of n oscillators with the states of an assemblage of identical particles, each having an n-dimensional state space, then the statement that the coordinates do not commute but obey (1) becomes the statement that the particles are not bosons but have creation operators $a_i^\dagger$ satisfying

$$a_j^\dagger a_i^\dagger = q_{ij} a_i^\dagger a_j^\dagger.$$

If the state label i becomes continuous and represents spatial position, so that we are dealing with field operators $\phi(x)$, then these will satisfy

$$\phi(x)\phi(y) = q(x, y)\phi(y)\phi(x)$$

i.e. not Bose statistics but *local anyon* statistics.

The dual form. Let V^* be the n-dimensional vector space spanned by $x_1, \ldots, x_n$ and V its dual. We think of the elements of V^* as coordinate functions on V, so that the algebra A generated by V^* is a coordinate algebra on V [8]. Equations (3) and (4), restricted to V^*, define a representation $\rho : G \otimes V^* \to V^*$ of our quantum group G

in which, with respect to the basis $x_1, \ldots, x_n$, the generators D_i, E_i and F_i are represented by elementary diagonal, subdiagonal and superdiagonal matrices respectively. We denote by a_{ij} the matrix elements of this representation, regarded as functions on G, i.e. elements of the dual G^*; then

$$\begin{aligned} \langle a_{ij}, D_k \rangle &= \delta_{ik}\delta_{jk} \\ \langle a_{ij}, E_k \rangle &= \delta_{i,k+1}\delta_{jk} \\ \langle a_{ij}, F_k \rangle &= \delta_{ik}\delta_{j,k+1} \end{aligned} \tag{18}$$

where the angle brackets denote the pairing between G and G^*.

The transposes of these matrices yield $\rho^*: G \otimes V \to V$ which is an antirepresentation of G, i.e. a representation of the opposite algebra G^{op}. Dualising this gives a map $\delta: V^* \to G^* \otimes V^*$ given by

$$\delta(x_i) = \sum_k a_{ki} \otimes x_k$$

or, in matrix notation,

$$\delta(\boldsymbol{x}^{\mathrm{T}}) = \boldsymbol{x}^{\mathrm{T}} \otimes \boldsymbol{A}.$$

The statement that each $\rho(g)$ is a generalised derivation of A with the comultiplication Δ dualises to the statement that δ extends to a homomorphism $\delta: A \to G^* \otimes A$ (in the terminology of [8], the coordinate algebra A is compatible with the representation ρ^*). Hence (2) is satisfied by $\Sigma_k\, a_{ki} \otimes x_k$, and so

$$\sum_{kl} \frac{p_{ij}}{p_{kl}}\, a_{ki}a_{lj} \otimes y_{kl} = \sum_{kl} \frac{p_{ji}}{p_{lk}}\, a_{lj}a_{ki} \otimes y_{lk}$$

where $y_{kl} = p_{kl}x_kx_l$. Since these satisfy $y_{kl} = y_{lk}$ (see (2)) but are otherwise independent, it follows that

$$\frac{p_{ij}}{p_{kl}}\, a_{ki}a_{lj} + \frac{p_{ij}}{p_{lk}}\, a_{li}a_{kj} = \frac{p_{ji}}{p_{kl}}\, a_{kj}a_{li} + \frac{p_{ji}}{p_{lk}}\, a_{lj}a_{ki}. \tag{19}$$

These relations are compatible with matrix comultiplication

$$\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj} \tag{20}$$

(i.e. $\Delta(a_{ij})$ satisfy (19) if a_{ij} do) since this is the comultiplication in G^*, a_{ij} being matrix elements of a representation of G.

Equation (19) constitutes half of the relations obtained by Manin [1] for the matrices of operators on a quantum superspace in which all the coordinates are even. In order to generate a deformation of the algebra of polynomials in n^2 commuting variables, the a_{ij} must satisfy an equal number of further independent relations. Instead of the remaining relations postulated by Manin, our $a_{ij} \in G^*$ satisfy

$$\frac{p'_{ij}}{p'_{kl}}\, a_{ik}a_{jl} + \frac{p'_{ij}}{p'_{lk}}\, a_{il}a_{jk} = \frac{p'_{ji}}{p'_{kl}}\, a_{jk}a_{il} + \frac{p'_{ji}}{p'_{lk}}\, a_{jl}a_{ik} \tag{21}$$

where

$$p'_{ij} = \begin{cases} u^2 q_{ij}^{-1} & \text{if } i < j \\ 1 & \text{if } i \geqslant j. \end{cases} \tag{22}$$

These relations could not have been anticipated from the construction of the a_{ij}. To prove them, note that they are the conditions on the matrix $A=(a_{ij})$ for the map $x\to A\otimes x$ to preserve the relations $p'_{ji}x_jx_i=p'_{ij}x_ix_j$, and therefore they are compatible with the matrix comultiplication (20). It follows from this that to prove that they hold in G^* it is sufficient to verify them when bracketed with the generators D_i, E_i and F_i of G. This is readily done, using the brackets (18) and the coproducts (14) together with the definition

$$\langle a_{ij}a_{kl}, X\rangle=\langle a_{ij}\otimes a_{kl}, \Delta(X)\rangle \qquad \text{for } X\in G. \tag{23}$$

The relations (19) and (21) can also be related to algebras of q-anticommuting coordinates which are dual to the q-commuting coordinates considered above. The full set of conditions is as follows:

$$\begin{aligned}(19)&\Leftrightarrow x^{\top}\to x^{\top}A \text{ preserves } p_{ji}x_jx_i=p_{ij}x_ix_j\\ &\Leftrightarrow \xi\to A\xi \text{ preserves } p_{ij}\xi_j\xi_i=-p_{ji}\xi_i\xi_j \text{ and } \xi_i^2=0 \end{aligned}\tag{24}$$

$$\begin{aligned}(21)&\Leftrightarrow x\to Ax \text{ preserves } p'_{ji}x_jx_i=p'_{ij}x_ix_j\\ &\Leftrightarrow \xi^{\top}\to \xi^{\top}A \text{ preserves } p'_{ij}\xi_j\xi_i=-p'_{ji}\xi_i\xi_j \text{ and } \xi_i^2=0. \end{aligned}\tag{25}$$

These co-actions of G^* on algebras of q-anticommuting coordinates ξ and ξ' make it possible to define determinants and adjugates and hence to give a formula for the antipode in G^* which is dual to (17). There are in fact two determinants in G^*, one arising from the row conditions (19) and one from the column conditions (21). Let Ξ be the algebra generated by $\xi_1,\ldots,\xi_n$ with relations as in (24), and let $\delta:\Xi\to G^*\otimes\Xi$ be the comultiplication indicated there:

$$\delta(\xi_i)=\sum_j a_{ij}\otimes\xi_j.$$

Similarly, let Ξ' be the algebra generated by $\xi'_1,\ldots,\xi'_n$ with relations as in (25), and let $\delta':\Xi'\to\Xi'\otimes G^*$ be given by

$$\delta(\xi'_i)=\sum_j \xi'_j\otimes a_{ji}.$$

Then δ and δ' both extend to algebra homomorphisms which are corepresentations of G^*. In both Ξ and Ξ' the subspace of homogeneous elements of degree n is one-dimensional, so we can define a row determinant D and a column determinant D' by

$$\delta(\xi_1\ldots\xi_n)=D\otimes\xi_1\ldots\xi_n$$

and

$$\delta'(\xi'_1\ldots\xi'_n)=D'\otimes\xi'_1\ldots\xi'_n.$$

It follows from the corepresentation property that D and D' are multiplicative:

$$\Delta(D)=D\otimes D \qquad \Delta(D')=D'\otimes D'.$$

Explicitly, they are given by

$$D=\sum_{\rho\in S_n}\varepsilon_G(\rho)a_{1,\rho(n)}\ldots a_{n,\rho(n)} \tag{26}$$

and

$$D=\sum_{\rho\in S_n}\varepsilon'_G(\rho)a_{\rho(1),1}\ldots a_{\rho(n),n} \tag{27}$$

where the sums are over all permutations ρ of $\{1, \ldots, n\}$ and the quantum signatures ε_G and ε'_G are defined by

$$\xi_{\rho(1)} \ldots \xi_{\rho(n)} = \varepsilon_G(\rho)\xi_1 \ldots \xi_n$$

and similarly for ε'_G. This can be expressed in a formula as follows:

$$\varepsilon_G(\rho) = \prod_{i<j} \frac{1}{p_{\rho(j),\rho(i)}} \left[\frac{\rho(i)-\rho(j)}{i-j}\right]. \tag{28}$$

Now define the cofactors A_{ij} and A'_{ij} by

$$A_{ij} = \sum_{\rho(i)=j} \frac{\varepsilon_G(\rho\circ\sigma_i)}{\varepsilon_G(\sigma_i)} a_{1,\rho(1)} \ldots]a_{ij}[\ldots a_{n,\rho(n)}$$

and

$$A'_{ij} = \sum_{\rho(j)=i} \frac{\varepsilon_G(\rho\circ\sigma'_j)}{\varepsilon_G(\sigma'_j)} a_{\rho(1),1} \ldots]a_{ij}[\ldots a_{\rho(n),n}$$

where σ_i and σ'_j denote the cyclic permutations

$$\sigma_i = (i \ldots 1) \qquad \sigma'_j = (j \ldots n)$$

and the notation $\ldots]x[\ldots$ indicates that x is to be omitted. Then by considering the tensor product with $\xi_1 \ldots \xi_n$ and using the homomorphism property (24), one can show that

$$\sum_j a_{ij}A_{kj} = D\delta_{ik}.$$

Similarly,

$$\sum_j A'_{ji}a_{jk} = D'\delta_{ik}.$$

It follows that the antipode in G^* is given by

$$S^*(a_{ij}) = A_{ji}D^{-1} = D'^{-1}A'_{ji}. \tag{29}$$

The difference between this algebra and that of [1] is that in the latter p'_{ij} is taken to be equal to p_{ij}. However, it can be shown that they must be related by (22), for some value of u, for the algebra defined by (19) and (21) to have the following consistency property.

The relations (19) and (21) refer to a 2×2 submatrix of A. Suppose $i<j$ and $k<l$, and write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$$

$$q = q_{ij} \qquad p = u^2 q_{kl}^{-1}.$$

Then (19) and (21) can be written as

$$\begin{aligned} ba &= pab & db &= qbd \\ ca &= qac & dc &= pcd \\ cb &= qp^{-1}bc & & \\ da &= u^{-2}pqad + (1-u^{-2})qbc. & & \end{aligned} \tag{30}$$

These relations enable any monomial in the a_{ij} to be expressed as a sum of lexicographically ordered monomials. The re-ordering procedure is not unique, but Manin [1] has shown that the different possible procedures will lead to the same result for all monomials if they do so for cubic monomials, when there are just two possible ways of reordering zyx to xyz:

$$\begin{array}{ccccccc} & & zxy & \longrightarrow & xzy & & \\ & \nearrow & & & & \searrow & \\ zyx & & & & & & xyz. \\ & \searrow & & & & \nearrow & \\ & & yzx & \longrightarrow & yxz & & \end{array} \qquad (31)$$

By considering all possible relative positions of the three elements x, y, z in the matrix $\boldsymbol{A}$, it can be verified that the two ways of re-ordering zyx, using (30), give the same result in all cases. It follows that the lexicographically ordered monomials are independent and form a basis of G^*.

When $n=2$ the three parameters p, q, u are related by $u^2=pq$ and the last relation of (30) becomes

$$da-ad=qbc-q^{-1}cb. \qquad (32)$$

The relations (30) can now be expressed in the form

$$\sum_{kl} R_{ij,kl}a_{km}a_{ln}=\sum_{kl} a_{jl}a_{ik}R_{kl,mn} \qquad (33)$$

with the R-matrix

$$\boldsymbol{R}=\begin{bmatrix} pq & 0 & 0 & 0 \\ 0 & q & pq-1 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & pq \end{bmatrix} \qquad (34)$$

(with rows and columns labelled by ij and kl in the order 11, 12, 21, 22). This satisfies the Yang–Baxter equation.

If $q=p$ this gives the familiar one-parameter deformation [2, 4] of $\mathbf{gl}(2)$. If $q=p^{-1}$ the R-matrix is in factorised form

$$\boldsymbol{R}=\boldsymbol{Q}\otimes\boldsymbol{Q}^{-1} \qquad \text{where} \qquad \boldsymbol{Q}=\begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

and the relations in the algebra G^* can be written as

$$a_{kl}a_{ij}=q^{i-j-k+l}a_{ij}a_{kl}.$$

For all values of p and q the algebra defined by (30) and (32) has the power property noted by Corrigan *et al* [9, 10]: if the matrix elements of $\boldsymbol{A}$ satisfy these relations, then those of $\boldsymbol{A}^n$ satisfy similar relations with (q,p) replaced by (q^n,p^n).

For general n, the usual one-parameter deformation [2, 4] of gl(n) is obtained by taking all the q_{ij} equal to q for $i<j$, and $u=q$. The resulting algebra is quasitriangular, i.e. is described by an R-matrix which satisfies the Yang–Baxter equation. This is also true for general q_{ij} if $u=1$, when the relations (19) and (21) become

$$q_{ij}a_{ik}a_{jl}=q_{kl}a_{jl}a_{ik} \qquad (35)$$

which is the same as (33) with the R-matrix

$$R_{ij,kl}=q_{ij}\delta_{ik}\delta_{jl}.$$

If q_{ij} factorises as $q_{ij} = q_i q_j^{-1}$ to give an R-matrix $\boldsymbol{R} = \boldsymbol{Q} \otimes \boldsymbol{Q}^{-1}$, then the $n \times n$ matrix $\boldsymbol{A}$ has the power property that the elements of $\boldsymbol{A}^n$ satisfy (35) with q_{ij} replaced by q_{ij}^n.

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